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On the sum of the reciprocals of k-generalized Fibonacci numbers

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Abstract

In this note, we that if $\{F_n^{(k)}\}_{n\ge 0}$ denotes the k-generalized Fibonacci sequence then for $n \ge 2$ the closest integer to the reciprocal of $\sum_{m\ge n} 1/F_m^{(k)}$ is $F_n^{(k)} - F_{n-1}^{(k)}$.

1 The problem and the result

There are many papers in the literature which address the integer part of the reciprocal of the sum $\sum_{m\ge n} 1/U_m$, where $\{U_n\}_{n\ge 1}$ is a binary recurrent sequence of positive integers. For example, the case of the Fibonacci sequence was treated by Ohstuka and Nakamura [4], the case of the Pell sequence was treated by Zhang and Wang [5], and the more general case of Lucas sequences of characteristic equation $x^2 - ax - 1$ with an integer $a \ge 1$ (which includes the particular case of the Fibonacci sequence for $a = 1$ and Pell sequence for $a = 2$) was treated in [3]. Letting $\{U_n\}_{n\ge 0}$ be this last Lucas sequence given by $U_0 = 0$, $U_1 = 1$ and $U_{n+2} = aU_{n+1} + U_n$ for all $n \ge 0$, one of the main results of [3] is that for $n \ge 1$

$$\left\lfloor \left(\sum_{m\ge n} \frac{1}{U_m} \right)^{-1} \right\rfloor = U_n - U_{n-1} - \delta_n$$

where $\delta_n = 0$ if n is even and $\delta_n = 1$ if n is odd.

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Here, for an integer $k \geq 2$, we prove a result of the same flavour for the kth order recurrent sequence $\{F_n^{(k)}\}_{n\geq -(k-2)}$ given by $F_i^{(k)} = 0$ for $i = -(k-2),\ -(k-3),\ \ldots,\ 0$ and $F_1^{(k)} = 1$ and

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2.$$

This sequence coincides with the Fibonacci sequence for $k = 2$. For any real number x let $\lfloor x \rceil$ be the closest integer to x (when x is at distance $1/2$ from an integer we can pick for $\lfloor x \rceil$ to be anyone of $\lfloor x \rfloor$ or $\lfloor x \rfloor + 1$). Our theorem is the following.

Theorem 1. *For $k \geq 2$ and $n \geq 2$, we have*

$$\left\lfloor \left(\sum_{m\geq n} \frac{1}{F_m^{(k)}} \right)^{-1} \right\rceil = F_n^{(k)} - F_{n-1}^{(k)}. \tag{1}$$

Many sequences naturally arising in nature and engineering are modelled by $\{F_n^{(k)}\}_{n\geq 0}$ for some $k \geq 2$. For a fixed k, as a linearly recurrent sequence, $F_n^{(k)}$ has a Binet formula. It turns out that this Binet formula has one term corresponding to the *dominant root* (see the next section for formal definitions), and then $F_n^{(k)}$ is the closest integer to this term. Let ε_n be the error of this approximation (formally, this also depends on k but we will omit the dependence on k in order not to clutter the exposition). The proof is then achieved by approximating the left–hand side of (1) with a natural candidate arising from the sum of the reciprocals of a certain geometric progression and relating the error of this approximation to $|\varepsilon_n|$ and $|\varepsilon_n - \varepsilon_{n-1}|$. Then the proof is completed by giving good upper bounds on $|\varepsilon_n|$ and $|\varepsilon_n - \varepsilon_{n-1}|$. The proof uses some ideas from [2].

2 Preliminary results on k-generalized Fibonacci numbers

It is known that the characteristic polynomial of the k–generalized Fibonacci numbers $F^{(k)} := (F_m^{(k)})_{m\geq 2-k}$, namely

$$\Psi_k(x) := x^k - x^{k-1} - \cdots - x - 1,$$

has just one root outside the unit circle. Let $\alpha := \alpha(k)$ denote that single root, which is located between $2\left(1 - 2^{-k}\right)$ and 2 (see [2]). To simplify notation, in our application we shall omit the dependence on k of α. We shall use $\alpha_1, \ldots, \alpha_k$ for all roots of $\Psi_k(x)$ with the convention that $\alpha_1 := \alpha$.

We now consider for an integer $k \geq 2$, the function

$$f_k(z) = \frac{z-1}{2+(k+1)(z-2)} \qquad \text{for} \qquad z \in \mathbb{C}. \tag{2}$$

With this notation, Dresden and Du presented in [2] the following "Binet–like" formula for the terms of $F^{(k)}$:

$$F_m^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i){\alpha_i}^{m-1}. \tag{3}$$

It was proved in [2] that the contribution of the roots which are inside the unit circle to the formula (3) is very small, namely that the approximation

$$\left|F_m^{(k)} - f_k(\alpha)\alpha^{m-1}\right| < \frac{1}{2} \quad \text{holds for all} \quad m \geqslant 2-k. \tag{4}$$

It was proved by Bravo and Luca in [1] that

$$\alpha^{m-2} \leq F_m^{(k)} \leq \alpha^{m-1} \qquad \text{holds for all} \qquad m \geq 1 \quad \text{and} \quad k \geq 2. \tag{5}$$

The root α is called the *dominant root* of $\{F_m^{(k)}\}_{m\geq-(k-2)}$. It is also known, and it will be useful for us, that

$$F_n^{(k)} = 2^{n-2} \quad \text{holds for all} \quad n \in [2, k+1],$$

whereas $F_{k+2}^{(k)} = 2^k - 1$.

Before we conclude this section, we present one more some useful lemma which was proved by Bravo and Luca in [1].

Lemma 1. *Let $k \geq 2$, α be the dominant root of $\{F_m^{(k)}\}_{m\geq-(k-2)}$, and consider the function $f_k(z)$ defined in* (2)*. Then*

$$\frac{1}{2} < f_k(\alpha) < \frac{3}{4}.$$

3 Two Lemmas

We put $\varepsilon_n := F_n - f_k(\alpha)\alpha^{n-1}$ for $n \geq -(k-2)$. As we mentioned in Section 2, in [2] the following result was proved.

Lemma 2. *We have $|\varepsilon_n| < 1/2$ for all $n \geq -(k-2)$.*

This lemma was proved in the following way. First it was checked that it folds for $n \in [-(k-2), 1]$, an interval containing k consecutive integers. Then since

$$\varepsilon_n = \sum_{i=2}^{n} f_k(\alpha_i)\alpha_i^n,$$

where $\alpha_2, \ldots, \alpha_k$ are all the other roots of $\Psi_k(X)$ which are complex numbers inside the unit circle, it follows that $\varepsilon_n \to 0$ as $n \to \infty$. Using the recurrence relation

$$\varepsilon_{n+1} = 2\varepsilon_n - \varepsilon_{n-k}$$

valid for all $n \geq 2$, it was then shown that the fact that $|\varepsilon_n| < 1/2$ for $n \in [-(k-2), 1]$ implies that $|\varepsilon_n| < 1/2$ for all $n \geq -(k-2)$. Here is a slight generalisation of that result.

Lemma 3. *Let $N_0 \geq -(k-2)$ be an integer, and $\{\delta_n\}_{n \geq N_0}$ be a sequence of real numbers whose Binet formula is given by*

$$\delta_n = \sum_{i=2}^{k} c_i \alpha_i^n \qquad \text{for all} \qquad n \geq N_0. \tag{6}$$

Assume that there are $n_0 \geq N_0$ and λ such that $|\delta_n| < \lambda$ holds for all $n \in [n_0, n_0 + k - 1]$. Then $|\delta_n| < \lambda$ holds for all $n \geq n_0$.

Proof. Formula (6) shows that δ_n tends to 0 as n tends to infinity. Also, the same formula shows that

$$\delta_{n+1} = 2\delta_n - \delta_{n-k} \qquad \text{holds for all} \qquad n \geq N_0 + k \tag{7}$$

since recurrence (7) is a consequence of the Binet formula (6). Assume that there is $n_1 \geq n_0$ such that $|\delta_{n_1}| \geq \lambda$ and let n_1 be minimal with this property. Clearly, $n_1 \geq n_0 + k$. Then the recurrence (7) in $n = n_1$ gives

$$\delta_{n_1+1} = 2\delta_{n_1} - \delta_{n_1-k}$$

and shows that $|\delta_{n_1+1}| \geq 2|\delta_{n_1}| - |\delta_{n_1-k}| \geq |\delta_{n_1}|$. By the same argument, we then get that $|\delta_{n_1+2}| \geq |\delta_{n_1+1}|$. This pattern continues by the same argument, so we get $|\delta_{n+1}| \geq |\delta_n|$ for all $n \geq n_1$, which contradicts the fact that δ_n tends to 0. Thus, $|\delta_n| < \lambda$ must always hold whenever $n \geq n_0$. $\square$

4 The proof modulo two estimates

The first part of the proof consists of evaluating the sum of a geometric series and keeping track of the errors of approximation. Since $F_1^{(k)} = F_2^{(k)} = 1$ and

$F_n^{(k)} \le 2^{n-2}$ holds for all $n \ge 2$ and the inequality is strict for $n > k+1$, we have that

$$2 = \sum_{m\ge 2} \frac{1}{2^{m-2}} < \sum_{m\ge 2} \frac{1}{F_n^{(k)}}.$$

This shows that

$$\left(\sum_{m\ge 2} \frac{1}{F_m^{(k)}}\right)^{-1} < \frac{1}{2},$$

therefore formula (1) holds for $n = 2$ (both its sides are 0). From now on, we assume that $n \ge 3$. We recall

$$F_m^{(k)} = f_k(\alpha)\alpha^{m-1} + \varepsilon_m \qquad \text{for} \qquad m \ge -(k-2),$$

where $|\varepsilon_m| < 1/2$ for all $m \ge -(k-2)$ by Lemma 2. We also put

$$\lambda_n := \max_{m\ge n} |\varepsilon_m|.$$

We then have

$$\begin{aligned}
\sum_{m\ge n} \frac{1}{F_m^{(k)}} &= \sum_{m\ge n} \frac{1}{f_k(\alpha)\alpha^{m-1}} + \sum_{m\ge n}\left(\frac{1}{F_m^{(k)}} - \frac{1}{f_k(\alpha)\alpha^{m-1}}\right)\\
&:= \frac{1}{f_k(\alpha)\alpha^{n-1}}\left(\sum_{j\ge 0}\frac{1}{\alpha^j}\right) + T_n = \frac{1}{f_k(\alpha)\alpha^{n-1}(1-1/\alpha)} + T_n.
\end{aligned}$$

We estimate $|T_n|$. We have, using estimate (5),

$$\begin{aligned}
|T_n| &= \left|\sum_{m\ge n}\frac{f_k(\alpha)\alpha^{m-1} - F_m^{(k)}}{f_k(\alpha)\alpha^{m-1}F_m^{(k)}}\right| = \left|-\sum_{m\ge n}\frac{\epsilon_m}{fk(\alpha)\alpha^{m-1}F_m^{(k)}}\right|\\
&\le \sum_{m\ge n}\frac{|\varepsilon_m|}{f_k(\alpha)\alpha^{m-1}F_m^{(k)}} \le \lambda_n \sum_{m\ge n}\frac{1}{f_k(\alpha)\alpha^{m-1}F_m^{(k)}}\\
&\le \lambda_n \sum_{m\ge n}\frac{1}{f_k(\alpha)\alpha^{2m-3}} = \frac{\lambda_n}{f_k(\alpha)\alpha^{2n-3}}\sum_{j\ge 0}\frac{1}{\alpha^{2j}}\\
&\le \frac{\lambda_n}{f_k(\alpha)\alpha^{2n-3}(1-1/\alpha^2)}.
\end{aligned}$$

Thus,

$$\sum_{m\ge n}\frac{1}{F_m^{(k)}} = \frac{1}{f_k(\alpha)\alpha^{n-1}(1-1/\alpha)}\left(1+\eta_n\right),$$

where

$$|\eta_n| = |T_n| f_k(\alpha)\alpha^{n-1}(1-1/\alpha) \le \frac{\lambda_n}{\alpha^{n-2}(1+1/\alpha)}. \tag{8}$$

Since $k \ge 2$, $\alpha \ge (1+\sqrt{5})/2 > 1.6$ and $n \ge 3$, we have that $\alpha^{n-2}(\alpha+1) > 4$ so the above upper bound is at most $1/8$. Thus,

$$\left(\sum_{m\ge n} \frac{1}{F_m^{(k)}}\right)^{-1} = f_k(\alpha)\alpha^{n-1}(1-1/\alpha)(1+\eta_n)^{-1}.$$

We use

$$(1+\eta_n)^{-1} = 1 - \eta_n + \eta_n^2 - \cdots,$$

which is valid on our range for η_n. Putting $\zeta_n := (1+\eta_n)^{-1} - 1$, we have, by (8), that

$$\begin{aligned} |\zeta_n| &= |\eta_n||1-\eta_n+\eta_n^2-\cdots| \\ &\le |\eta_n|\left(1+\left(\frac{\lambda_n}{\alpha^{n-2}(1+1/\alpha)}\right)+\left(\frac{\lambda_n}{\alpha^{n-2}(1+1/\alpha)}\right)^2+\cdots\right) \\ &= \frac{|\eta_n|}{1-\lambda_n/(\alpha^{n-2}(1+1/\alpha))} \le \frac{\lambda_n}{\alpha^{n-2}(1+1/\alpha)-\lambda_n}. \end{aligned}$$

Hence,

$$\begin{aligned} \left(\sum_{m\ge n} \frac{1}{F_m^{(k)}}\right)^{-1} &= f_k(\alpha)\alpha^{n-1} - f_k(\alpha)\alpha^{n-2} + \big(f_k(\alpha)\alpha^{n-1}(1-1/\alpha)\zeta_n\big) \\ &:= F_n^{(k)} - F_{n-1}^{(k)} - \varepsilon_n + \varepsilon_{n-1} + \delta_n, \end{aligned} \tag{9}$$

where

$$\begin{aligned} |\delta_n| &= \big|f_k(\alpha)\alpha^{n-1}(1-1/\alpha)\zeta_n\big| \le \frac{f_k(\alpha)\alpha^{n-1}(1-1/\alpha)\lambda_n}{\alpha^{n-2}(1+1/\alpha)-\lambda_n} \\ &= \frac{f_k(\alpha)(\alpha-1)\lambda_n}{1+1/\alpha-\lambda_n/\alpha^{n-2}} < \frac{3\lambda_n}{5}. \end{aligned}$$

The last inequality holds because $f_k(\alpha) < 3/4$ (by Lemma 1), $\alpha - 1 < 1$, therefore

$$f_k(\alpha)(\alpha-1) < 3/4,$$

while

$$1 + 1/\alpha - \lambda_n/\alpha^{n-2} \ge 1 + (1-\lambda_n)/\alpha > 1 + 1/(2\alpha) > 5/4,$$

where we used the fact that $n \geq 3$ and $\lambda_n < 1/2$. Assume that

$$|\varepsilon_n - \varepsilon_{n-1}| + \frac{3\lambda_n}{5} < \frac{1}{2}. \tag{10}$$

Then

$$|-\varepsilon_n + \varepsilon_{n-1} + \delta_n| \leq |\varepsilon_n - \varepsilon_{n-1}| + |\delta_n| < |\varepsilon_n - \varepsilon_{n-1}| + \frac{3\lambda_n}{5} < \frac{1}{2},$$

so, by estimate (9), we get

$$\left\lfloor \left(\sum_{m \geq n} \frac{1}{F_m^{(k)}} \right)^{-1} \right\rfloor = F_n^{(k)} - F_{n-1}^{(k)}.$$

This finishes the proof of the theorem modulo proving the following lemma.

Lemma 4. *The estimates*

$$\lambda_{n-1} < \frac{1}{3.2} \quad \textit{and} \quad |\varepsilon_n - \varepsilon_{n-1}| < \frac{1}{3.2} \tag{11}$$

hold for all $n \geq 3$ and $k \geq 3$.

Note that if (11) holds, then since $\lambda_n \leq \lambda_{n-1}$, we have that

$$\left| \varepsilon_n - \varepsilon_{n-1} + \frac{3\lambda_n}{5} \right| \leq |\varepsilon_n - \varepsilon_{n-1}| + \frac{3\lambda_{n-1}}{5} < \frac{1}{3.2}\left(1 + \frac{3}{5}\right) = \frac{1}{2},$$

so (11) implies (10) and therefore the conclusion of the theorem for n.

5 The proof of the estimates: Lemma 4

Let us start with $k = 2$. In this case, $\alpha = (1 + \sqrt{5})/2$, and $\varepsilon_n = -\beta^n/\sqrt{5}$, where $\beta = -\alpha^{-1}$ is the conjugate of α. Thus, for $n \geq 2$, we have

$$|\varepsilon_n| = \frac{1}{\sqrt{5}\alpha^n} \leq \frac{1}{\sqrt{5}\alpha^2} < 0.18 < \frac{1}{3.2}.$$

Furthermore, for $n \geq 3$, we have

$$|\varepsilon_n - \varepsilon_{n-1}| = \frac{|\beta|^{n-1}(1-\beta)}{\sqrt{5}} = \frac{1}{\sqrt{5}\alpha^{n-2}} \leq \frac{1}{\sqrt{5}\alpha} < 0.28 < \frac{1}{3.2}.$$

From now on, we assume that $k \geq 3$. For what follows, we will need a slightly better approximation of α than the mere fact that $\alpha \in (2(1 - 1/2^k), 2)$.

Lemma 5. *We have*

$$\alpha = 2 - \frac{1}{2^k} - \frac{c_k}{2^{2k-2}}, \quad \textit{where} \quad c_k \in (0,k).$$

Proof. We check that the above estimate holds for $k = 2,3$. For $k \geq 4$, we note that α satisfies the equation

$$0 = \alpha^k - \alpha^{k-1} - \cdots - 1 = \alpha^k - \frac{\alpha^k - 1}{\alpha - 1} = \frac{\alpha^{k+1} - 2\alpha^k + 1}{\alpha - 1}.$$

Thus,

$$\alpha = 2 - \frac{1}{\alpha^k}.$$

Now $\alpha = 2(1 - \zeta/2^k)$, where $\zeta \in (0,1)$. Thus,

$$\alpha = 2 - \frac{1}{2^k}\left(1 - \frac{\zeta}{2^k}\right)^{-k} = 2 - \frac{1}{2^k}\exp\left(-k\log\left(1 - \frac{\zeta}{2^k}\right)\right).$$

Using that for $x \in (0,1/2)$ we have $\log(1-x) = -y$ for some $y \in (0,2x)$, we get that $-\log(1-\zeta/2^k) = \eta$, where $\eta \in (0,1/2^{k-1})$. Thus, $k\eta \in (0,k/2^{k-1})$ and $k/2^{k-1} \leq 1/2$ for $k \geq 4$. Using that $\exp y = 1 + z$ for some $z \in (0,2y)$ if $y \in (0,1/2)$, we have that

$$\exp(-k\log(1-\zeta/2^k)) = \exp(k\eta) = 1 + \delta, \qquad \text{where} \qquad \delta \in (0,k/2^{k-2}).$$

Thus, writing $\delta := c_k/2^{k-2}$, we have that $c_k \in (0,k)$ and

$$\alpha = 2 - \frac{1}{2^k}\left(1 + \frac{c_k}{2^{k-2}}\right) = 1 - \frac{1}{2^k} - \frac{c_k}{2^{2k-2}},$$

which is what we wanted. □

In order to prove that (11) holds in the ranges indicated by Lemma 4 it suffices, by Lemma 3 with $\delta_n := \varepsilon_n$ or $\delta_n := \varepsilon_n - \varepsilon_{n-1}$, $\lambda := 1/3.2$ and $n_0 := 3$, to show that inequality (11) holds for the first k values of the ranges indicated in (1) and (2) of Lemma 4. Let's get to work.

Lemma 6. *We have for $n \in [2, k+1]$,*

$$\varepsilon_n = \frac{n-k}{2^{k+3-n}}\left(1 + \frac{c_k}{2^{k-2}}\right) + \delta_{n,k}, \quad \textit{with} \quad |\delta_{n,k}| < \frac{(k+1)^2}{2^{k-3}}\left(1 + \frac{k}{2^{k-2}}\right)^2. \tag{12}$$

Proof. We use the fact that for $n \in [2, k+1]$, we have for

$$g_{k,n}(z) := f_k(z) z^{n-1}, \qquad \text{that} \qquad g_{k,n}(2) = 2^{n-2} = F_n^{(k)}.$$

Thus,

$$\varepsilon_n = F_n^{(k)} - g_{k,n}(\alpha) = g_{k,n}(2) - g_{k,n}(\alpha) = g'_{k,n}(2)(2-\alpha) - \frac{1}{2} g''_{k,n}(\zeta)(\alpha - 2)^2 \tag{13}$$

for some $\zeta \in (\alpha, 2)$, a formula which is obtained by applying the Taylor formula to the expansion of $g_{k,n}(z)$ around $z = 2$. Now

$$g_{k,n}(z) = \frac{z^n}{2 + (k+1)(z-2)} - \frac{z^{n-1}}{2 + (k+1)(z-2)},$$

so

$$\begin{aligned} g'_{k,n}(z) &= \frac{n z^{n-1}}{2 + (k+1)(z-2)} - \frac{(n-1) z^{n-2}}{2 + (k+1)(z-2)} \\ &- \frac{(k+1) z^n}{(2 + (k+1)(z-2))^2} + \frac{(k+1) z^{n-1}}{(2 + (k+1)(z-2))^2}. \end{aligned} \tag{14}$$

Evaluating the above in $n = 2$, we get

$$\begin{aligned} g'_{k,n}(2) &= n 2^{n-2} - (n-1) 2^{n-3} - (k+1) 2^{n-2} + (k+1) 2^{n-3} \\ &= 2^{n-3}(2n - (n-1) - 2(k+1) + k + 1) = (n-k) 2^{n-3}. \end{aligned}$$

Thus,

$$g'_{k,n}(2)(2-\alpha) = \frac{(n-k) 2^{n-3}}{2^k} \left(1 + \frac{c_k}{2^{k-2}}\right) = \frac{n-k}{2^{k+3-n}} \left(1 + \frac{c_k}{2^{k-2}}\right). \tag{15}$$

This is the main term. For the next term, we take again the derivative of $g'_{k,n}$ given by formula (14). This formula consists in 8 fractions and we evaluate them in $\zeta \in (\alpha, 2)$. The largest numerator is $(k+1)^2 \zeta^n < (k+1)^2 \cdot \zeta^{k+1}$. Since $\alpha - 2 \geq -1/2^{k-1}$, the denominator is at least

$$2 + (k+1)(\alpha - 2) \geq 2 - \frac{k+1}{2^{k-1}} \geq 1 \qquad \text{for} \qquad k \geq 3.$$

Hence,

$$|g''_{k,n}(\zeta)| < 8(k+1)^2 \zeta^{k+1} < (k+1)^2 2^{k+4}.$$

Since

$$(\alpha - 2)^2 = \frac{1}{2^{2k}} \left(1 + \frac{c_k}{2^{k-1}}\right)^2$$

by Lemma 5, we get that

$$|\delta_{n,k}| \le \frac{(k+1)^2 2^{k+4}}{2^{2k+1}} \left(1+\frac{c_k}{2^{k-2}}\right)^2 = \frac{(k+1)^2}{2^{k-3}} \left(1+\frac{k}{2^{k-2}}\right)^2. \tag{16}$$

The proof follows from (13), (15) and (16). □

Proof of Lemma 4. For $n = k$, the main term in 0 in (12). For $n = k-1$, the fraction $|n-k|/2^{3+k-n}$ evaluates to $1/16$. For $n \le k-2$, putting $x := k-n \ge 1$, the fraction $|n-k|/2^{3+k-n}$ equals $x/2^{3+x}$, a function which is decreasing for $x \ge 2$, so its maximal value is at $x = 2$ and equals again $1/16$. The worst case scenario for $n \in [2, k+1]$ is therefore in $n = k+1$, for which the fraction $|n-k|/2^{3+n-k}$ evaluates to $1/4$. We thus get that for $n \in [2, k+1]$, we have that

$$|\varepsilon_n| \le \frac{1}{16}\left(1+\frac{k}{2^{k-2}}\right) + \frac{(k+1)^2}{2^{k-3}}\left(1+\frac{k}{2^{k-2}}\right)^2 \quad \text{for} \quad n \in [2, k],$$

and

$$|\varepsilon_{k+1}| < \frac{1}{4}\left(1+\frac{k}{2^{k-2}}\right) + \frac{(k+1)^2}{2^{k-3}}\left(1+\frac{k}{2^{k-2}}\right)^2.$$

The right–hand sides above are $< 1/3.2$ for $k \ge 20$. In particular, we have that $|\varepsilon_n| \le 1/3.2$ for all $n \in [2, k+1]$ if $k \ge 20$, and by Lemma 3, $|\varepsilon_n| \le 1/3.2$ for all $n \ge 2$.

We now consider

$$\delta_n := \varepsilon_n - \varepsilon_{n-1} \quad \text{for} \quad n \in [3, k+2].$$

By the above arguments, for $n \in [3, k]$, we have that

$$|\delta_n| \le |\varepsilon_n| + |\varepsilon_{n-1}| \le \frac{1}{8}\left(1+\frac{k}{2^{k-2}}\right) + \frac{(k+1)^2}{2^{k-4}}\left(1+\frac{k}{2^{k-2}}\right)^2. \tag{17}$$

For $n = k+1$, we have

$$|\delta_{k+1}| = |\varepsilon_{k+1} - \varepsilon_k| \le \frac{1}{4}\left(1+\frac{k}{2^{k-2}}\right) + \frac{(k+1)^2}{2^{k-4}}\left(1+\frac{k}{2^{k-2}}\right)^2, \tag{18}$$

where we used the fact that at $n = k$ the main term of ε_n in (12) equals 0. For $n = k+2$, we have

$$\varepsilon_{k+2} = 2\varepsilon_{k+1} - \varepsilon_1,$$

so

$$|\varepsilon_{k+2} - \varepsilon_{k+1}| = |\varepsilon_{k+1} - \varepsilon_1|.$$

Now
$$\varepsilon_{k+1}=\frac{1}{4}\left(1+\frac{c_k}{2^{k-2}}\right)+\delta_{k,k+1},$$
while
$$\varepsilon_1=1-f_k(\alpha)=1-(f_k(2)+f_k'(\zeta)(\alpha-2))=\frac{1}{2}-f_k'(\zeta)(\alpha-2)\quad\text{for}\quad\zeta\in(\alpha,2).$$
Clearly,
$$\begin{aligned}|f_k'(\zeta)| &= \left|\frac{2+(k+1)(\zeta-2)-(k+1)(\zeta-1)}{2+(k+1)(\zeta-2))^2}\right|\\ &= \frac{k-1}{(2+(k+1)(\zeta-2))^2}<k-1.\end{aligned}$$
Thus,
$$\begin{aligned}|\delta_{k+2}| &= |\zeta_{k+1}-\zeta_1|\\ &\le \frac{1}{4}+\frac{c_k}{2^k}+|\delta_{k,k+1}|+|f_k'(\zeta)|(2-\alpha)\\ &< \frac{1}{4}+\frac{k}{2^k}+\frac{(k+1)^2}{2^{k-3}}\left(1+\frac{k}{2^{k-2}}\right)^2+\frac{k-1}{2^k}\left(1+\frac{k}{2^{k-2}}\right). \quad (19)\end{aligned}$$

For $k\ge 20$, all right–hand sides of (17), (18) and (19) are $<1/3.2$. Thus, $|\varepsilon_n-\varepsilon_{n-1}|<1/3.2$ holds for all $n\in[3,k+2]$, a interval of length k. By Lemma 3, it holds for all $n\ge 3$.

A computer program now checked that $|\varepsilon_n|<1/3.2$ also holds for all $k\in[3,19]$ and all $n\in[2,k+1]$. For this, we just computed
$$|2^{n-2}-f_k(\alpha)\alpha^{n-1}|\quad\text{for all}\quad k\in[3,19]\quad\text{and}\quad n\in[2,k+1].$$
In fact, the maximum value of $|\varepsilon_n|$ in this range was less than $0.24996<1/3.2$. Similarly, we checked that $|\varepsilon_n-\varepsilon_{n-1}|<1/3.2$ holds for all $n\in[3,k+2]$ and all $k\in[3,19]$. The way we did it was to compute, for all $n\in[3,k+1]$, the amount
$$|\varepsilon_n-\varepsilon_{n-1}|=|2^{n-2}-2^{n-3}-f_k(\alpha)(\alpha-1)\alpha^{n-2}|,$$
and to check that it is $<1/3.2$ in this range. When $n=k+2$, the term $2^{n-2}-2^{n-3}=2^{n-3}$ must be replaced by $2^{n-3}-1$ because for this n, we have $F_n^{(k)}=2^{n-2}-1$. The maximal value of $|\varepsilon_n-\varepsilon_{n-1}|$ in this range was less than $0.261<1/3.2$.

The theorem is proved.

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